

ALGEBRAIC CURVES

EXERCISE SHEET 6

Unless otherwise specified, k is an algebraically closed field.

Exercise 1.

Let V, W be varieties and assume that W is affine.

- (1) Show that there is a bijection $\text{Hom}_{\text{Var}}(V, W) \simeq \text{Hom}_{k\text{-alg}}(\mathcal{O}(W), \mathcal{O}(V))$.
- (2) Show that there is a bijection $\mathcal{O}(V) \simeq \text{Hom}_{\text{Var}}(V, \mathbb{A}_k^1)$.
- (3) Show that $\mathcal{O}(\mathbb{P}_k^1) \simeq k$.
- (4) Show that $\mathcal{O}(\mathbb{P}_k^n) \simeq k$ for all $n \geq 1$.

Exercise 2.

Let $n \geq 1$ and $f \in k[x_1, \dots, x_n]$.

- (1) Show that $\mathbb{A}_k^n - V(f)$ is affine. What is its ring of regular functions?
- (2) Show that $\mathbb{A}_k^2 - \{(0, 0)\}$ is not affine. (Hint: compute the ring of regular functions).

Exercise 3.

Let $\varphi : V \rightarrow W$ be a morphism of affine varieties and $\varphi^\# : \Gamma(W) \rightarrow \Gamma(V)$ the corresponding morphism of coordinate rings. Let $P \in V$ and $Q = \varphi(P)$ and consider local rings $\mathcal{O}_P(V)$, $\mathcal{O}_Q(W)$ with maximal ideals $\mathfrak{m}_P, \mathfrak{m}_Q$. Show that $\varphi^\#$ extends uniquely to a ring homomorphism $\mathcal{O}_Q(W) \rightarrow \mathcal{O}_P(V)$ and that $\varphi^\#(\mathfrak{m}_Q) \subseteq \mathfrak{m}_P$.

Exercise 4.

Let $n \geq 1$ and V a variety. We use projective coordinates x_i , $1 \leq i \leq n+1$ on $\mathbb{P}_k^n$. Suppose there exist an open cover $(U_i)_{1 \leq i \leq n+1}$ of V and morphisms of varieties $\varphi_i : U_i \rightarrow \{x_i \neq 0\} \subseteq \mathbb{P}_k^n$, $1 \leq i \leq n+1$, such that $\forall i \neq j$, $(\varphi_i)_{|U_i \cap U_j} = (\varphi_j)_{|U_i \cap U_j}$. Show that there exists a unique morphism $\varphi : V \rightarrow \mathbb{P}_k^n$ such that $\varphi_{|U_i} = \varphi_i$. We say that φ is obtained by *glueing* the φ_i , $1 \leq i \leq n+1$.

Exercise 5. *

Let $f \in k[x_1, x_2, x_3]$ an irreducible form of degree 2 and consider $V_P(f) \subseteq \mathbb{P}_k^2$.

- (1) Show that, up to a linear change of coordinates, we can assume that $f = x_2^2 - x_1x_3$. (Hint: remember we classified similar subvarieties of $\mathbb{A}_k^2$).
- (2) Show that the map:

$$\begin{array}{ccc} \mathbb{P}_k^1 & \rightarrow & \mathbb{P}_k^2 \\ (s : t) & \mapsto & (s^2 : st : t^2) \end{array}$$

induces an isomorphism $\mathbb{P}_k^1 \simeq V_P(f)$. (Hint: take a look locally in the standard affine opens of projective space and use exercise 4).